Bravais colourings of planar modules with N-fold symmetry

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Abstract. The first step in investigating colour symmetries for periodic and aperiodic systems is the determination of all colouring schemes that are compatible with the symmetry group of the underlying structure, or with a subgroup of it. For an important class of colourings of planar structures, this mainly combinatorial question can be addressed with methods of algebraic number theory. We present the corresponding results for all planar modules with N-fold symmetry that emerge as the rings of integers in cyclotomic fields with class number one. The counting functions are multiplicative and can be encapsulated in Dirichlet series generating functions, which turn out to be the Dedekind zeta functions of the corresponding cyclotomic fields.

Key Words: Colourings, Planar Modules, Cyclotomic Fields, Dirichlet Series

1 Introduction

Colour symmetries of crystals [23, 24, 21, 22, 19] and, more recently, of quasicrystals [16, 13, 1] continue to attract a lot of attention, simply because so little is known about their classification, see [13] for a recent review. A first step in this analysis consists in answering the question of how many different colourings of an infinite point set exist which are compatible with its underlying symmetry. More precisely, one has to determine the possible numbers of colours, and to count the corresponding possibilities to distribute the colours over the point set (up to permutations), in line with all compatible symmetry constraints.

In this generality, the problem has not even been solved for simple lattices. One common restriction is to demand that one colour occupies a subset which is of the same Bravais type as the original set, while the other colours encode the cosets. Of particular interest are the cases where the point symmetry is irreducible. In this situation, to which we will also restrict ourselves, several results are known and can be given in closed form [1, 5, 6, 13, 14, 19].

Particularly interesting are planar cases because, on the one hand, they show up in quasicrystalline T-phases, and, on the other hand, they are linked to the rather interesting classification of planar Bravais classes with n-fold symmetry [15], which is based on a connection to algebraic number theory in general, and to cyclotomic fields in particular. The Bravais types correspond to ideal classes and are unique for the following 29 choices of n,

$$n \;\in\; \{3,4,5,7,8,9,11,12,13,15,16,\\ 17,19,20,21,24,25,27,28,32,\\ 33,35,36,40,44,45,48,60,84\}. \tag{1}$$

The canonical representatives are the sets of cyclotomic integers $\mathcal{M}_n = \mathbb{Z}[\xi_n]$, the ring of polynomials in ξ_n, a primitive nth root of unity. To be explicit (which is not necessary), we choose $\xi_n = \exp(2\pi\mathrm{i}/n)$. Apart from $n = 1$ (where $\mathcal{M}_1 = \mathbb{Z}$ is one-dimensional), the values of n in (1) correspond to all cases where $\mathbb{Z}[\xi_n]$ is a principal ideal domain and thus has class number one, see [26, 6] for details. If n is odd, we have $\mathcal{M}_n = \mathcal{M}_{2n}$. Consequently, $\mathcal{M}_n$ has N-fold symmetry where

$$N \;=\; N(n) \;=\; \begin{cases} n & \text{if } n \text{ is even,} \\ 2n & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

To avoid duplication of results, values $n \equiv 2 \bmod 4$ do not appear in the above list (1). There are systematic mathematical reasons to prefer this convention to the notation of [13, 15], some of which will appear later on.

It is this very connection to algebraic number theory which results in the Bravais classification [15], and also allows for a solution of the combinatorial part of the colouring problem by means of Dirichlet series generating functions, compare [1, 6]. The values of n from our list (1) are naturally grouped according to $\phi(n)$, which is Euler's totient function

$$\phi(n) \;=\; \{1 \le k \le n \mid \gcd(k,n) = 1\}\,. \tag{3}$$

Note that $\phi(n) = 2$ covers the two crystallographic cases $n = 3$ (triangular lattice) and $n = 4$ (square lattice), while $\phi(n) = 4$ means $n \in \{5, 8, 12\}$ which are the standard symmetries of genuine planar quasicrystals. Again, $n = 10$ is covered implicitly, as explained above.

The methods emerging from the connection to number theory are useful for the description of aperiodic order in general [17]. In the planar case, the link to cyclotomic fields allows the full treatment of all 29 cases listed above, which was observed and used in several papers [15, 18, 1, 6]. Here,

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we shall explain the use of cyclotomic fields more explicitly, and present detailed results for the combinatorial part of the colouring problem in all 29 cases. We will not follow the standard approach of algebraic number theory (see [8] for a good and readable general introduction), but take the slightly unusual point of view of Dirichlet series generating functions. This allows for a rather straightforward calculation of all quantities required, starting from the representation theory of finite Abelian groups. Furthermore, we shall also present some asymptotic results that can be extracted from the generating function.

The modules arise in quasicrystal theory in several ways. One is through the Fourier module that supports the Bragg peaks. Another, more important, is via the limit translation module of a (discrete) quasiperiodic tiling. Characteristic points of the latter (e.g., vertex points) are then model sets (or cut and project sets) on the basis of the entire (dense) module, seen as a lattice in a space of higher dimension, see [1, 3, 5] for details. This way, one obtains a one-to-one relation between colourings of discrete objects and of the underlying dense module. Since the latter is universal, it is the natural object to study.

Previously, results for the cases with $\phi(n) \leq 10$ were given in [1, 13, 3, 4], mainly without proofs. Other combinatorial problems of (quasi)crystallography have been addressed by similar methods, compare [2].

2 Number theoretic formulation

In view of the above remarks, we formulate the problem immediately in terms of the full modules. Consider $\mathcal{M}_n$ with a fixed n, e.g., from our list (1). Then, $\mathcal{M}_n$ is an Abelian group, and also a $\mathbb{Z}$-module of rank $\phi(n)$. We view it as a subset of the Euclidean plane, identified with $\mathbb{C}$, and hence as a geometric object. In this setting, important subgroups of $\mathcal{M}_n$ are those that are images of $\mathcal{M}_n$ under a similarity transformation, i.e., a mapping

$$z \mapsto az \quad \text{or} \quad z \mapsto a\bar{z} \qquad \text{with } a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}\,.$$

Such subgroups are called *similarity submodules*, see [6] for details. Similarity submodules are very natural objects algebraically, and have been studied in the context of colouring problems in dimensions two, three and four [1, 6, 7]. To make this more precise, let us start with a definition.

Definition 1 *A* Bravais colouring *of the module $\mathcal{M}_n$ with k colours is a mapping $c\colon \mathcal{M}_n \to \{1, 2, \ldots, k\}$ such that one of the sets $c^{-1}(\ell)$, for $1 \leq \ell \leq k$, is a similarity submodule of $\mathcal{M}_n$ of index k, and the others, one by one, are the corresponding cosets in $\mathcal{M}_n$. The number of Bravais colourings of $\mathcal{M}_n$ with k colours, up to permutations of the colours, is denoted by $a_n(k)$.*

In other words, a Bravais colouring c of $\mathcal{M}_n$ is a partition

$$\mathcal{M}_n \;=\; \dot{\bigcup_{1\leq \ell\leq k}} c^{-1}(\ell) \tag{4}$$

into k disjoint sets that are translates of each other and carry pairwise different colours. In addition, one of the sets is a similarity submodule of $\mathcal{M}_n$. The interest in this kind of partition originates from its rather important group-theoretical structure, compare [21, 22, 23, 24, 13] and references therein. Also, each such partition induces a colouring of module compatible tilings (usually formulated in terms of local indistinguishability, compare [13]), see [3, 5, 20] for examples.

Here, we concentrate on the combinatorial aspect, i.e., on the determination of the values of $a_n(k)$, which define an integer-valued arithmetic function.

Lemma 1 *Let $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ with $n \geq 3$. A similarity submodule of $\mathcal{M}_n$ is a principal ideal of $\mathbb{Z}[\xi_n]$. Consequently, $a_n(k)$ is the number of principal ideals of $\mathbb{Z}[\xi_n]$ of norm k.*

PROOF. Let us observe that the module $\mathbb{Z}[\xi_n]$ is invariant under complex conjugation. Consequently, all similarity submodules of $\mathbb{Z}[\xi_n]$ are of the form $a\mathbb{Z}[\xi_n]$ with $a \in \mathbb{C}^*$. As $a\mathbb{Z}[\xi_n] \subset \mathbb{Z}[\xi_n]$, we must have $0 \neq a \in \mathbb{Z}[\xi_n]$. So, a similarity submodule is a principal ideal, and vice versa. Each principal ideal gives rise to precisely one Bravais colouring c of $\mathbb{Z}[\xi_n]$, up to permutations of the colours, where, without loss of generality, $c^{-1}(1)$ is the principal ideal. The norm of an ideal equals the subgroup index $k = \bigl[\mathbb{Z}[\xi_n] : a\mathbb{Z}[\xi_n]\bigr]$, hence counts the number of cosets, resp. colours. In view of Eq. (4), and the meaning of the sets $c^{-1}(\ell)$, our assertion follows. □

For n from the list (1), *all* ideals of $\mathbb{Z}[\xi_n]$ are principal, because the corresponding cyclotomic fields $\mathbb{Q}(\xi_n)$ have class number one [26, Thm. 11.1]. This list is exhaustive except for the case $n = 1$ which corresponds to the field $\mathbb{Q}$ itself. Our combinatorial problem then amounts to counting all (non-zero) ideals of a given index. Consequently, the arithmetic function $a_n(k)$ is *multiplicative*, i.e., $a_n(k\ell) = a_n(k)a_n(\ell)$ for k, ℓ coprime, and $a_n(1) = 1$. A succinct way to encapsulate the numbers $a_n(k)$ is the use of a Dirichlet series generating function [27],

$$F_n(s) \;:=\; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s}\,. \tag{5}$$

As we will see, this function converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The connection to the cyclotomic fields admits the direct calculation of this Dirichlet series by means of algebraic number theory.

Proposition 1 *For all n in the list* (1)*, the Dirichlet series $F_n(s)$ of Eq.* (5) *equals the Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\xi_n)$, i.e.,*

$$F_n(s) \;=\; \zeta_{\mathbb{Q}(\xi_n)}(s) \;:=\; \sum_{\mathfrak{a}} \frac{1}{\mathrm{norm}(\mathfrak{a})^s}\,, \quad \mathrm{Re}(s) > 1\,,$$

where $\mathfrak{a}$ runs over all non-zero ideals of $\mathbb{Z}[\xi_n]$.

PROOF. The Dedekind zeta function of $\mathbb{Q}(\xi_n)$ is, by definition, the Dirichlet series generating function of the number of ideals of a given index in the maximal order $\mathbb{Z}[\xi_n]$. Since n is from our list (1), we are in the class number one case, hence all ideals are principal, and the claim follows from Lemma 1. The convergence result is standard [26]. □

Table 1. Basic indices for the unramified primes of $\mathbb{Z}[\xi_n]$ with n from (1). The symbol $\llcorner\frac{\ell}{k}\urcorner$ means that primes $p \equiv k \bmod n$ contribute via p^ℓ as basic index, where ℓ is the smallest integer such that $k^\ell \equiv 1 \bmod n$, and the integer m entering Eq. (7) is $m = \phi(n)/\ell$.

$\phi(n)$	n	general primes p
2	3	$\frac{1}{1}$ $\frac{2}{2}$
	4	$\frac{1}{1}$ $\frac{2}{3}$
4	5	$\frac{1}{1}$ $\frac{4}{2}$ $\frac{4}{3}$ $\frac{2}{4}$
	8	$\frac{1}{1}$ $\frac{2}{3}$ $\frac{2}{5}$ $\frac{2}{7}$
	12	$\frac{1}{1}$ $\frac{2}{5}$ $\frac{2}{7}$ $\frac{2}{11}$
6	7	$\frac{1}{1}$ $\frac{3}{2}$ $\frac{6}{3}$ $\frac{3}{4}$ $\frac{6}{5}$ $\frac{2}{6}$
	9	$\frac{1}{1}$ $\frac{6}{2}$ $\frac{3}{4}$ $\frac{6}{5}$ $\frac{3}{7}$ $\frac{2}{8}$
8	15	$\frac{1}{1}$ $\frac{4}{2}$ $\frac{2}{4}$ $\frac{4}{7}$ $\frac{4}{8}$ $\frac{2}{11}$ $\frac{4}{13}$ $\frac{2}{14}$
	16	$\frac{1}{1}$ $\frac{4}{3}$ $\frac{4}{5}$ $\frac{2}{7}$ $\frac{2}{9}$ $\frac{4}{11}$ $\frac{4}{13}$ $\frac{2}{15}$
	20	$\frac{1}{1}$ $\frac{4}{3}$ $\frac{4}{7}$ $\frac{2}{9}$ $\frac{2}{11}$ $\frac{4}{13}$ $\frac{4}{17}$ $\frac{2}{19}$
	24	$\frac{1}{1}$ $\frac{2}{5}$ $\frac{2}{7}$ $\frac{2}{11}$ $\frac{2}{13}$ $\frac{2}{17}$ $\frac{2}{19}$ $\frac{2}{23}$
10	11	$\frac{1}{1}$ $\frac{10}{2}$ $\frac{5}{3}$ $\frac{5}{4}$ $\frac{5}{5}$ $\frac{10}{6}$ $\frac{10}{7}$ $\frac{10}{8}$ $\frac{5}{9}$ $\frac{2}{10}$
12	13	$\frac{1}{1}$ $\frac{12}{2}$ $\frac{3}{3}$ $\frac{6}{4}$ $\frac{4}{5}$ $\frac{12}{6}$ $\frac{12}{7}$ $\frac{4}{8}$ $\frac{3}{9}$ $\frac{6}{10}$ $\frac{12}{11}$ $\frac{2}{12}$
	21	$\frac{1}{1}$ $\frac{6}{2}$ $\frac{3}{4}$ $\frac{6}{5}$ $\frac{2}{8}$ $\frac{6}{10}$ $\frac{6}{11}$ $\frac{2}{13}$ $\frac{3}{16}$ $\frac{6}{17}$ $\frac{6}{19}$ $\frac{2}{20}$
	28	$\frac{1}{1}$ $\frac{6}{3}$ $\frac{6}{5}$ $\frac{3}{9}$ $\frac{6}{11}$ $\frac{2}{13}$ $\frac{2}{15}$ $\frac{6}{17}$ $\frac{6}{19}$ $\frac{6}{23}$ $\frac{3}{25}$ $\frac{2}{27}$
	36	$\frac{1}{1}$ $\frac{6}{5}$ $\frac{6}{7}$ $\frac{6}{11}$ $\frac{3}{13}$ $\frac{2}{17}$ $\frac{2}{19}$ $\frac{6}{23}$ $\frac{3}{25}$ $\frac{6}{29}$ $\frac{6}{31}$ $\frac{2}{35}$
16	17	$\frac{1}{1}$ $\frac{8}{2}$ $\frac{16}{3}$ $\frac{4}{4}$ $\frac{16}{5}$ $\frac{16}{6}$ $\frac{16}{7}$ $\frac{8}{8}$ $\frac{8}{9}$ $\frac{16}{10}$ $\frac{16}{11}$ $\frac{16}{12}$ $\frac{4}{13}$ $\frac{16}{14}$ $\frac{8}{15}$ $\frac{2}{16}$
	32	$\frac{1}{1}$ $\frac{8}{3}$ $\frac{8}{5}$ $\frac{4}{7}$ $\frac{4}{9}$ $\frac{8}{11}$ $\frac{8}{13}$ $\frac{2}{15}$ $\frac{2}{17}$ $\frac{8}{19}$ $\frac{8}{21}$ $\frac{4}{23}$ $\frac{4}{25}$ $\frac{8}{27}$ $\frac{8}{29}$ $\frac{2}{31}$
	40	$\frac{1}{1}$ $\frac{4}{3}$ $\frac{4}{7}$ $\frac{2}{9}$ $\frac{2}{11}$ $\frac{4}{13}$ $\frac{4}{17}$ $\frac{2}{19}$ $\frac{2}{21}$ $\frac{4}{23}$ $\frac{4}{27}$ $\frac{2}{29}$ $\frac{2}{31}$ $\frac{4}{33}$ $\frac{4}{37}$ $\frac{2}{39}$
	48	$\frac{1}{1}$ $\frac{4}{5}$ $\frac{2}{7}$ $\frac{4}{11}$ $\frac{4}{13}$ $\frac{2}{17}$ $\frac{4}{19}$ $\frac{2}{23}$ $\frac{2}{25}$ $\frac{4}{29}$ $\frac{2}{31}$ $\frac{4}{35}$ $\frac{4}{37}$ $\frac{2}{41}$ $\frac{4}{43}$ $\frac{2}{47}$
	60	$\frac{1}{1}$ $\frac{4}{7}$ $\frac{2}{11}$ $\frac{4}{13}$ $\frac{4}{17}$ $\frac{2}{19}$ $\frac{4}{23}$ $\frac{2}{29}$ $\frac{2}{31}$ $\frac{4}{37}$ $\frac{2}{41}$ $\frac{4}{43}$ $\frac{4}{47}$ $\frac{2}{49}$ $\frac{4}{53}$ $\frac{2}{59}$
18	19	$\frac{1}{1}$ $\frac{18}{2}$ $\frac{18}{3}$ $\frac{9}{4}$ $\frac{9}{5}$ $\frac{9}{6}$ $\frac{3}{7}$ $\frac{6}{8}$ $\frac{9}{9}$ $\frac{18}{10}$ $\frac{3}{11}$ $\frac{6}{12}$ $\frac{18}{13}$ $\frac{18}{14}$ $\frac{18}{15}$ $\frac{9}{16}$ $\frac{9}{17}$ $\frac{2}{18}$
	27	$\frac{1}{1}$ $\frac{18}{2}$ $\frac{9}{4}$ $\frac{18}{5}$ $\frac{9}{7}$ $\frac{6}{8}$ $\frac{3}{10}$ $\frac{18}{11}$ $\frac{9}{13}$ $\frac{18}{14}$ $\frac{9}{16}$ $\frac{6}{17}$ $\frac{3}{19}$ $\frac{18}{20}$ $\frac{9}{22}$ $\frac{18}{23}$ $\frac{9}{25}$ $\frac{2}{26}$
20	25	$\frac{1}{1}$ $\frac{20}{2}$ $\frac{20}{3}$ $\frac{10}{4}$ $\frac{5}{6}$ $\frac{4}{7}$ $\frac{20}{8}$ $\frac{10}{9}$ $\frac{5}{11}$ $\frac{20}{12}$ $\frac{20}{13}$ $\frac{10}{14}$ $\frac{5}{16}$ $\frac{20}{17}$ $\frac{4}{18}$ $\frac{10}{19}$ $\frac{5}{21}$ $\frac{20}{22}$ $\frac{20}{23}$ $\frac{2}{24}$
	33	$\frac{1}{1}$ $\frac{10}{2}$ $\frac{5}{4}$ $\frac{10}{5}$ $\frac{10}{7}$ $\frac{10}{8}$ $\frac{2}{10}$ $\frac{10}{13}$ $\frac{10}{14}$ $\frac{5}{16}$ $\frac{10}{17}$ $\frac{10}{19}$ $\frac{10}{20}$ $\frac{2}{23}$ $\frac{5}{25}$ $\frac{10}{26}$ $\frac{10}{28}$ $\frac{10}{29}$ $\frac{5}{31}$ $\frac{2}{32}$
	44	$\frac{1}{1}$ $\frac{10}{3}$ $\frac{5}{5}$ $\frac{10}{7}$ $\frac{5}{9}$ $\frac{10}{13}$ $\frac{10}{15}$ $\frac{10}{17}$ $\frac{10}{19}$ $\frac{2}{21}$ $\frac{2}{23}$ $\frac{5}{25}$ $\frac{10}{27}$ $\frac{10}{29}$ $\frac{10}{31}$ $\frac{10}{35}$ $\frac{5}{37}$ $\frac{10}{39}$ $\frac{10}{41}$ $\frac{2}{43}$
24	35	$\frac{1}{1}$ $\frac{12}{2}$ $\frac{12}{3}$ $\frac{6}{4}$ $\frac{2}{6}$ $\frac{4}{8}$ $\frac{6}{9}$ $\frac{3}{11}$ $\frac{12}{12}$ $\frac{4}{13}$ $\frac{3}{16}$ $\frac{12}{17}$ $\frac{12}{18}$ $\frac{6}{19}$ $\frac{4}{22}$ $\frac{12}{23}$ $\frac{6}{24}$ $\frac{6}{26}$ $\frac{4}{27}$ $\frac{2}{29}$ $\frac{6}{31}$ $\frac{12}{32}$ $\frac{12}{33}$ $\frac{2}{34}$
	45	$\frac{1}{1}$ $\frac{12}{2}$ $\frac{6}{4}$ $\frac{12}{7}$ $\frac{4}{8}$ $\frac{6}{11}$ $\frac{12}{13}$ $\frac{6}{14}$ $\frac{3}{16}$ $\frac{4}{17}$ $\frac{2}{19}$ $\frac{12}{22}$ $\frac{12}{23}$ $\frac{2}{26}$ $\frac{4}{28}$ $\frac{6}{29}$ $\frac{3}{31}$ $\frac{12}{32}$ $\frac{6}{34}$ $\frac{4}{37}$ $\frac{12}{38}$ $\frac{6}{41}$ $\frac{12}{43}$ $\frac{2}{44}$
	84	$\frac{1}{1}$ $\frac{6}{5}$ $\frac{6}{11}$ $\frac{2}{13}$ $\frac{6}{17}$ $\frac{6}{19}$ $\frac{6}{23}$ $\frac{3}{25}$ $\frac{2}{29}$ $\frac{6}{31}$ $\frac{3}{37}$ $\frac{2}{41}$ $\frac{2}{43}$ $\frac{6}{47}$ $\frac{6}{53}$ $\frac{2}{55}$ $\frac{6}{59}$ $\frac{6}{61}$ $\frac{6}{65}$ $\frac{6}{67}$ $\frac{2}{71}$ $\frac{6}{73}$ $\frac{6}{79}$ $\frac{2}{83}$

For general n, the arithmetic function $a_n(k)$ still counts the principal ideals of $\mathbb{Q}(\xi_n)$. Whenever non-principal ideals exist, $F_n(s)$ is no longer given by the Dedekind zeta function of $\mathbb{Q}(\xi_n)$. For the related problem of coincidence site modules, an example ($n = 23$) is treated in [18].

The multiplicativity of the arithmetic function $a_n(k)$ for n in the list (1) implies that it is sufficient to know the values $a_n(p^r)$ for all primes p and powers $r > 0$. On the level of the generating function, this corresponds to the Euler product expansion of $F_n(s)$, again for $\{\mathrm{Re}(s) > 1\}$,

$$F_n(s) \;:=\; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s} \;=\; \prod_p E_n(p^{-s}) \qquad (6)$$

where p runs over the primes of $\mathbb{Z}$.

As we will see below, each Euler factor is of the form

$$\begin{aligned} E_n(p^{-s}) &= \frac{1}{(1-p^{-\ell s})^m} \\ &= \sum_{j=0}^{\infty} \binom{j+m-1}{m-1} \frac{1}{(p^s)^{\ell j}} \end{aligned} \qquad (7)$$

from which one extracts the values of $a_n(p^r)$ for $r \geq 0$. The integers ℓ and m are characteristic quantities that depend on n and on the prime p. In particular, m is the number of prime ideal divisors of p, and ℓ is their degree (also called the residue class degree) [26]. With the usual approach of algebraic number theory, its determination (and that of the corresponding m) depends on whether p divides n or not. This can be circumvented with a formula based on Dirichlet characters, as we will explain below.

Table 2. List of all ramified primes with corresponding integers ℓ and m for $\mathbb{Z}[\xi_n]$ with n from the list (1). Here, r is the p-free part of n, and $\ell\, m = \phi(r)$. Details on the connection to Table 1 are given in the text.

$\phi(n)$	n	p	r	$\phi(r)$	ℓ	m
2	3	3	1	1	1	1
	4	2	1	1	1	1
4	5	5	1	1	1	1
	8	2	1	1	1	1
	12	2	3	2	2	1
		3	4	2	2	1
6	7	7	1	1	1	1
	9	3	1	1	1	1
8	15	3	5	4	4	1
		5	3	2	2	1
	16	2	1	1	1	1
	20	2	5	4	4	1
		5	4	2	1	2
	24	2	3	2	2	1
		3	8	4	2	2
10	11	11	1	1	1	1
12	13	13	1	1	1	1
	21	3	7	6	6	1
		7	3	2	1	2
	28	2	7	6	3	2
		7	4	2	2	1
	36	2	9	6	6	1
		3	4	2	2	1
16	17	17	1	1	1	1
	32	2	1	1	1	1
	40	2	5	4	4	1
		5	8	4	2	2
	48	2	3	2	2	1
		3	16	8	4	2
	60	2	15	8	4	2
		3	20	8	4	2
		5	12	4	2	2
18	19	19	1	1	1	1
	27	3	1	1	1	1
20	25	5	1	1	1	1
	33	3	11	10	5	2
		11	3	2	2	1
	44	2	11	10	10	1
		11	4	2	2	1
24	35	5	7	6	6	1
		7	5	4	4	1
	45	3	5	4	4	1
		5	9	6	6	1
	84	2	21	12	6	2
		3	28	12	6	2
		7	12	4	2	2

If p and n are coprime, we have $p \equiv k \bmod n$ for some k such that k and n are coprime, i.e., $\gcd(k, n) = 1$. In this case, $\ell\, m = \phi(n)$, which fixes m. Furthermore, the residue class degree ℓ is the smallest integer such that $k^{\ell} \equiv 1 \bmod n$, see [26, Thm. 2.3]. The cases of these primes are listed as $\llcorner\frac{\ell}{k}\lrcorner$ in Table 1.

In addition, for each n, there are finitely many primes p which divide n, the so-called *ramified* primes. In this case, $n = r\, p^t$ with $t \geq 1$ and r an integer not divisible by p, so that r is the p-free part of n. The values of ℓ and m needed in the case $p|n$ equal those needed for the situation where n is replaced by r, which brings us back to the previous case (the proof is more involved and can be extracted from [12]). In particular, $\ell\, m = \phi(r)$, and $\ell = m = 1$ whenever $r = 1$. The complete result is listed in Table 2 for convenience. With this information, one can easily calculate the possible numbers of colours and the generating functions by inserting (7) into (6) and expanding the Euler product, which is an easy task for an algebraic program package.

Let us briefly come back to Eq. (5). If $a_n(k)$ is multiplicative, the set of possible numbers of colours forms a semigroup with unit. It is generated by the basic indices which are obtained as p^{ℓ} from Tables 1 and 2. In more general situations, this semigroup structure is lost, compare the related coincidence problem [18].

3 Dirichlet characters and zeta functions

The purpose of this section is to summarise some basic properties of cyclotomic fields that are helpful to calculate the Dedekind zeta functions explicitly, and hence also the characteristic integers ℓ and m via Eq. (7), in a unified fashion. Moreover, some analytic properties of the zeta functions will become accessible this way, which we will need to determine asymptotic properties of our counting functions. The approach uses the well-known theory of Dirichlet characters, which we shall now summarise.

The object we start from is the *Galois group* G_n of the (cyclotomic) field extension $\mathbb{Q}(\xi_n)/\mathbb{Q}$. Here, the Galois group simply consists of all automorphisms σ of $\mathbb{Q}(\xi_n)$ that fix all rational elements, i.e., $\sigma(x) = x$ for all $x \in \mathbb{Q}$. It is an Abelian group of order $\phi(n)$. The following result is standard [26, Thm. 2.5].

Proposition 2 *The Galois group G_n of $\mathbb{Q}(\xi_n)/\mathbb{Q}$ is Abelian and of order $\phi(n)$. It is isomorphic to*

$$\{1 \leq k \leq n \mid \gcd(k, n) = 1\}$$

with multiplication modulo n as the group operation, and k representing the automorphism defined by the map $\xi_n \mapsto \xi_n^k$. For the cases with class number one, the group structure and a possible set of generators are as given in Table 3. □

Next, we need the Dirichlet characters, which can be seen as extensions of the characters χ of G_n in the sense of representation theory of (finite) Abelian groups. So, each χ is a group homomorphism from G_n into the unit circle in $\mathbb{C}$. With G_n as given in Proposition 2, χ is defined on all elements of $\{1 \leq k \leq n \mid \gcd(k, n) = 1\}$ modulo n. To obtain

Table 3. Galois groups G_n and their generators (written as residue classes $\bmod n$) for the 29 non-trivial cyclotomic fields with class number one, see Eq. (1).

n	$\phi(n)$	Galois group	generators
3	2	C_2	(2)
4	2	C_2	(3)
5	4	C_4	(2)
7	6	C_6	(3)
8	4	$C_2 \times C_2$	(3),(5)
9	6	C_6	(2)
11	10	C_{10}	(2)
12	4	$C_2 \times C_2$	(5),(7)
13	12	C_{12}	(2)
15	8	$C_4 \times C_2$	(2),(11)
16	8	$C_4 \times C_2$	(3),(7)
17	16	C_{16}	(3)
19	18	C_{18}	(2)
20	8	$C_4 \times C_2$	(3),(11)
21	12	$C_6 \times C_2$	(2),(13)
24	8	$C_2 \times C_2 \times C_2$	(5),(7),(13)
25	20	C_{20}	(2)
27	18	C_{18}	(2)
28	12	$C_6 \times C_2$	(3),(13)
32	16	$C_8 \times C_2$	(3),(15)
33	20	$C_{10} \times C_2$	(2),(10)
35	24	$C_{12} \times C_2$	(2),(6)
36	12	$C_6 \times C_2$	(5),(19)
40	16	$C_4 \times C_2 \times C_2$	(3),(11),(21)
44	20	$C_{10} \times C_2$	(3),(21)
45	24	$C_{12} \times C_2$	(2),(26)
48	16	$C_4 \times C_2 \times C_2$	(5),(17),(23)
60	16	$C_4 \times C_2 \times C_2$	(7),(11),(19)
84	24	$C_6 \times C_2 \times C_2$	(5),(13),(43)

a primitive Dirichlet character [26, Ch. 3], of which there are precisely $\phi(n)$ different ones, we have to extend χ to *all* integers. Clearly, χ is periodic with period n, so it suffices to define χ on the missing integers $\leq n$. This has to be done in such a way that the resulting character, which we still denote by χ, is totally multiplicative, i.e., $\chi(k\ell) = \chi(k)\chi(\ell)$ for *all* $k, \ell \geq 1$, and that the period of the resulting character is minimal. This minimal period, which is always a divisor of n, is called the *conductor* of χ, denoted by f_χ. In this process, one has $\chi(k) = 0$ whenever $\gcd(k, f_\chi) \neq 1$. An illustrative example ($n = 20$) is shown in Table 4.

Consider now a primitive Dirichlet character χ. Its L-series is defined as

$$L(s,\chi) \;:=\; \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} \;=\; \sum_{k=1}^{f_\chi} \chi(k) \sum_{j=0}^{\infty} \frac{1}{(k + j f_\chi)^s} \tag{8}$$

which has nice analytic properties. In particular, it converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, which follows by expressing the last sum in terms of the Hurwitz zeta function [26, Ch. 4].

Due to the total multiplicativity of χ, its L-series has a particularly simple Euler product expansion [26, p. 31], namely

$$L(s,\chi) \;=\; \prod_p \frac{1}{1-\chi(p)\, p^{-s}}\,, \quad \mathrm{Re}(s) > 1\,. \tag{9}$$

The following result is standard [26, Thm. 4.3].

Proposition 3 *The Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\xi_n)$ is given by*

$$\zeta_{\mathbb{Q}(\xi_n)}(s) \;=\; \prod_{\chi \in \widehat{G}_n} L(s,\chi)$$

where $\widehat{G}_n$ is the set of primitive Dirichlet characters of $\mathbb{Q}(\xi_n)/\mathbb{Q}$. $\square$

The Euler product expansion of $\zeta_{\mathbb{Q}(\xi_n)}(s)$ is

$$\begin{aligned} \zeta_{\mathbb{Q}(\xi_n)}(s) \;&=\; \prod_{\chi \in \widehat{G}_n} \prod_p \frac{1}{1-\chi(p)\, p^{-s}} \\ &=\; \prod_p \prod_{\chi \in \widehat{G}_n} \frac{1}{1-\chi(p)\, p^{-s}}\,, \end{aligned} \tag{10}$$

so, using Proposition 1, our Euler factors of Eq. (6) are

$$E_n(p^{-s}) \;=\; \prod_{\chi \in \widehat{G}_n} \bigl(1-\chi(p)\, p^{-s}\bigr)^{-1}. \tag{11}$$

Since χ is f_χ-periodic, and $f_\chi | n$, all primitive Dirichlet characters in $\widehat{G}_n$ are n-periodic, in agreement with their original construction. Therefore, the structure of an Euler factor can only depend on the residue class of $p \bmod n$.

Calculating the Euler factors $E_n(p^{-s})$, which means to expand the product on the right-hand side of Eq. (11), one finds that they are always of the form $(1 - p^{-\ell s})^{-m}$ with integers ℓ and m, compare Eq. (7). These integers, which are now simple arithmetic expressions in the values of the characters at p, are precisely the quantities introduced after Eq. (7). This works for all primes, and the distinction between $p|n$ and $p\nmid n$ is implicit, see Table 4 for an example.

Whenever a prime p is not ramified, which happens if and only if $\chi(p) \neq 0$ for all $\chi \in \widehat{G}_n$, these integers, as mentioned above, satisfy $\ell\, m = \phi(n)$. For the remaining primes, which are precisely those dividing n, the product ℓm is a true divisor of $\phi(n)$, and counts the number of characters not vanishing at p. In the example of Table 4, these are the primes $p = 2$ and $p = 5$, while all other primes are unramified. Further details, together with the meaning of ℓ and m for the splitting of primes in the algebraic field extension, can be found in [12, 26]. An explicit calculation can easily be done with an algebraic program package; a corresponding Mathematica® program for the calculation of the generating functions can be downloaded from [10].

We summarise the result as follows.

Theorem 1 *The Euler factors of the Dedekind zeta function of $\mathbb{Q}(\xi_n)$ are of the form given in Eq.* (7) *with characteristic indices ℓ and m that depend on p and n. Using the explicit representation via the L-series, the results for the class number one cases are those given in Tables* 1 *and* 2. $\square$

Table 4. Construction of all primitive Dirichlet characters for $n = 20$. The generators of $G_{20} \simeq C_4 \times C_2$ are $g = (3)$ and $h = (11)$. Each character χ originates from a product of characters of C_4 and C_2, and is thus labelled by a pair (i, j). The two extra lines show the characteristic indices ℓ and m for this example.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
χ	f_χ	e		g				g^3		g^2		h		gh				g^3h		g^2h	
$(0,0)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(1,0)$	5	1	$-\mathrm{i}$	i	-1	0	1	$-\mathrm{i}$	i	-1	0	1	$-\mathrm{i}$	i	-1	0	1	$-\mathrm{i}$	i	-1	0
$(2,0)$	5	1	-1	-1	1	0	1	-1	-1	1	0	1	-1	-1	1	0	1	-1	-1	1	0
$(3,0)$	5	1	i	$-\mathrm{i}$	-1	0	1	i	$-\mathrm{i}$	-1	0	1	i	$-\mathrm{i}$	-1	0	1	i	$-\mathrm{i}$	-1	0
$(0,1)$	20	1	0	1	0	0	0	1	0	1	0	-1	0	-1	0	0	0	-1	0	-1	0
$(1,1)$	20	1	0	i	0	0	0	$-\mathrm{i}$	0	-1	0	-1	0	$-\mathrm{i}$	0	0	0	i	0	1	0
$(2,1)$	4	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0
$(3,1)$	20	1	0	$-\mathrm{i}$	0	0	0	i	0	-1	0	-1	0	i	0	0	0	$-\mathrm{i}$	0	1	0
	ℓ	1	4	4		1		4		2		2		4				4		2	
	m	8	1	2		2		2		4		4		2				2		4	

Since these results have concrete applications in crystallography and materials science, we spell out the first few terms of the generating functions in Table 5. Explicit realisations of the corresponding colourings can be constructed by means of prime (ideal) factorisation in $\mathbb{Z}[\xi_n]$ and interpretation of the ideals as similarity submodules.

4 Asymptotic properties

Generating functions are an efficient way to encapsulate an entire series of numbers into one object. If the generating function is not only a formal object, but also has well-defined analytic properties, one can extract asymptotic properties of the sequence from the generating function. In the case of Dirichlet series, this requires some relatively advanced techniques from complex analysis. Fortunately, in the cases that emerged here, we are still in a rather simple situation, fully covered by the following special case of Delange's theorem, see [25, Ch. II.7, Thm. 15].

Proposition 4 *Let $F(s) = \sum_{k=1}^{\infty} a(k)\, k^{-s}$ be a Dirichlet series with non-negative coefficients which converges for $\mathrm{Re}(s) > 1$. Suppose that $F(s)$ is holomorphic at all points of the line $\{\mathrm{Re}(s) = 1\}$ except at $s = 1$. Approaching $s = 1$ from the half-plane $\{\mathrm{Re}(s) > 1\}$, let $F(s)$ have a singularity of the form $F(s) = g(s) + h(s)/(s-1)$, where both $g(s)$ and $h(s)$ are holomorphic at $s = 1$. Then, as $x \to \infty$,*

$$A(x) \;:=\; \sum_{k \le x} a(k) \;\sim\; h(1)\, x\, .$$

In other words, $h(1)$ is the asymptotic average value of $a(k)$ as $k \to \infty$. □

Our Dirichlet series $F_n(s)$ are Dedekind zeta functions of cyclotomic fields, according to Proposition 1. Then, Proposition 4 leads to the following explicit result.

Theorem 2 *Let n be a number from the list* (1). *The average number of Bravais colourings of $\mathcal{M}_n$, with a given number k of colours, is asymptotically, as $k \to \infty$, given by the residue of the Dedekind zeta function of $\mathbb{Q}(\xi_n)$ at $s = 1$. Consequently, for $x \to \infty$, one has*

$$A_n(x) \;:=\; \sum_{k \le x} a_n(k) \;\sim\; \alpha_n\, x\, ,$$

with

$$\alpha_n \;=\; \prod_{1 \not\equiv \chi \in \widehat{G}_n} L(1, \chi)\, .$$

Some values of α_n are given in Table 6.

PROOF. The Dedekind zeta functions of the cyclotomic fields (resp. their analytic continuations) always possess an isolated pole of first order at $s = 1$, and no other singularity in the entire complex plane, see [26, Ch. 4]. Consequently, by Proposition 4, $A_n(x)$ has linear growth as $x \to \infty$. The growth rate is then the residue of the zeta function at $s = 1$. The L-series $L(s, \chi)$ have analytic continuations to entire functions for all Dirichlet characters χ except for $\chi_0 \equiv 1$. In the latter case, one has $L(s, \chi_0) = \zeta(s)$. This is Riemann's zeta function which has a simple pole at $s = 1$ with residue 1. For all $\chi \neq \chi_0$, one has $L(1, \chi) \neq 0$, see [26, Cor. 4.4], and the expression for α_n is then immediate from Proposition 1. □

To apply this Theorem, we need to know how to calculate the values $L(1, \chi)$ for the non-trivial characters. This is described explicitly on p. 36 and in Thm. 4.9 of [26]. The result is

$$\alpha_n \;=\; \frac{R_n}{N(n)} \left(\frac{2\pi \prod_{p|n} p^{1/(p-1)}}{n} \right)^{\phi(n)/2}, \qquad (12)$$

where p in the product runs over all prime divisors of n, hence precisely over the ramified primes, $N(n)$ is the function of Eq. (2), and R_n is the *regulator* of the cyclotomic field $\mathbb{Q}(\xi_n)$. Its calculation, which is a bit technical, is explained in [26]. It is based on the knowledge of the fundamental units of the maximal order of $\mathbb{Q}(\xi_n + \bar{\xi}_n)$, the maximal real subfield of $\mathbb{Q}(\xi_n)$, followed by the calculation of a determinant [26, p. 41 and Prop. 4.16]. We show the first

Table 5. First terms of the Dirichlet series of Eq. (5) for the class number one cases as listed in (1).

n	$\zeta_{\mathcal{M}_n}(s)$
3	$1+\frac{1}{3^s}+\frac{1}{4^s}+\frac{2}{7^s}+\frac{1}{9^s}+\frac{1}{12^s}+\frac{2}{13^s}+\frac{1}{16^s}+\frac{2}{19^s}+\frac{2}{21^s}+\frac{1}{25^s}+\frac{1}{27^s}+\frac{2}{28^s}+\frac{2}{31^s}+\frac{1}{36^s}+\frac{2}{37^s}+\frac{2}{39^s}+\frac{2}{43^s}+\frac{1}{48^s}+\frac{3}{49^s}+\dots$
4	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{2}{5^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{2}{10^s}+\frac{2}{13^s}+\frac{1}{16^s}+\frac{2}{17^s}+\frac{1}{18^s}+\frac{2}{20^s}+\frac{3}{25^s}+\frac{2}{26^s}+\frac{2}{29^s}+\frac{1}{32^s}+\frac{2}{34^s}+\frac{1}{36^s}+\frac{2}{37^s}+\frac{2}{40^s}+\dots$
5	$1+\frac{1}{5^s}+\frac{4}{11^s}+\frac{1}{16^s}+\frac{1}{25^s}+\frac{4}{31^s}+\frac{4}{41^s}+\frac{4}{55^s}+\frac{4}{61^s}+\frac{4}{71^s}+\frac{1}{80^s}+\frac{1}{81^s}+\frac{4}{101^s}+\frac{10}{121^s}+\frac{1}{125^s}+\frac{4}{131^s}+\frac{4}{151^s}+\frac{4}{155^s}+\dots$
7	$1+\frac{1}{7^s}+\frac{2}{8^s}+\frac{6}{29^s}+\frac{6}{43^s}+\frac{1}{49^s}+\frac{2}{56^s}+\frac{3}{64^s}+\frac{6}{71^s}+\frac{6}{113^s}+\frac{6}{127^s}+\frac{3}{169^s}+\frac{6}{197^s}+\frac{6}{203^s}+\frac{6}{211^s}+\frac{12}{232^s}+\frac{6}{239^s}+\frac{6}{281^s}+\dots$
8	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{2}{9^s}+\frac{1}{16^s}+\frac{4}{17^s}+\frac{2}{18^s}+\frac{2}{25^s}+\frac{1}{32^s}+\frac{4}{34^s}+\frac{2}{36^s}+\frac{4}{41^s}+\frac{2}{49^s}+\frac{2}{50^s}+\frac{1}{64^s}+\frac{4}{68^s}+\frac{2}{72^s}+\frac{4}{73^s}+\frac{3}{81^s}+\dots$
9	$1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{6}{19^s}+\frac{1}{27^s}+\frac{6}{37^s}+\frac{6}{57^s}+\frac{1}{64^s}+\frac{6}{73^s}+\frac{1}{81^s}+\frac{6}{109^s}+\frac{6}{111^s}+\frac{6}{127^s}+\frac{6}{163^s}+\frac{6}{171^s}+\frac{6}{181^s}+\frac{1}{192^s}+\frac{6}{199^s}+\dots$
11	$1+\frac{1}{11^s}+\frac{10}{23^s}+\frac{10}{67^s}+\frac{10}{89^s}+\frac{1}{121^s}+\frac{10}{199^s}+\frac{2}{243^s}+\frac{10}{253^s}+\frac{10}{331^s}+\frac{10}{353^s}+\frac{10}{397^s}+\frac{10}{419^s}+\frac{10}{463^s}+\frac{55}{529^s}+\frac{10}{617^s}+\frac{10}{661^s}+\dots$
12	$1+\frac{1}{4^s}+\frac{1}{9^s}+\frac{4}{13^s}+\frac{1}{16^s}+\frac{2}{25^s}+\frac{1}{36^s}+\frac{4}{37^s}+\frac{2}{49^s}+\frac{4}{52^s}+\frac{4}{61^s}+\frac{1}{64^s}+\frac{4}{73^s}+\frac{1}{81^s}+\frac{4}{97^s}+\frac{2}{100^s}+\frac{4}{109^s}+\frac{4}{117^s}+\frac{2}{121^s}+\dots$
13	$1+\frac{1}{13^s}+\frac{4}{27^s}+\frac{12}{53^s}+\frac{12}{79^s}+\frac{12}{131^s}+\frac{12}{157^s}+\frac{1}{169^s}+\frac{12}{313^s}+\frac{4}{351^s}+\frac{12}{443^s}+\frac{12}{521^s}+\frac{12}{547^s}+\frac{12}{599^s}+\frac{3}{625^s}+\frac{12}{677^s}+\frac{12}{689^s}+\dots$
15	$1+\frac{2}{16^s}+\frac{1}{25^s}+\frac{8}{31^s}+\frac{8}{61^s}+\frac{1}{81^s}+\frac{4}{121^s}+\frac{8}{151^s}+\frac{8}{181^s}+\frac{8}{211^s}+\frac{8}{241^s}+\frac{3}{256^s}+\frac{8}{271^s}+\frac{8}{331^s}+\frac{4}{361^s}+\frac{2}{400^s}+\frac{8}{421^s}+\dots$
16	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{16^s}+\frac{8}{17^s}+\frac{1}{32^s}+\frac{8}{34^s}+\frac{4}{49^s}+\frac{1}{64^s}+\frac{8}{68^s}+\frac{2}{81^s}+\frac{8}{97^s}+\frac{4}{98^s}+\frac{8}{113^s}+\frac{1}{128^s}+\frac{8}{136^s}+\frac{2}{162^s}+\frac{8}{193^s}+\dots$
17	$1+\frac{1}{17^s}+\frac{16}{103^s}+\frac{16}{137^s}+\frac{16}{239^s}+\frac{2}{256^s}+\frac{1}{289^s}+\frac{16}{307^s}+\frac{16}{409^s}+\frac{16}{443^s}+\frac{16}{613^s}+\frac{16}{647^s}+\frac{16}{919^s}+\frac{16}{953^s}+\frac{16}{1021^s}+\frac{16}{1123^s}+\dots$
19	$1+\frac{1}{19^s}+\frac{18}{191^s}+\frac{18}{229^s}+\frac{6}{343^s}+\frac{1}{361^s}+\frac{18}{419^s}+\frac{18}{457^s}+\frac{18}{571^s}+\frac{18}{647^s}+\frac{18}{761^s}+\frac{18}{1103^s}+\frac{18}{1217^s}+\frac{6}{1331^s}+\frac{9}{1369^s}+\frac{18}{1483^s}+\dots$
20	$1+\frac{2}{5^s}+\frac{1}{16^s}+\frac{3}{25^s}+\frac{8}{41^s}+\frac{8}{61^s}+\frac{2}{80^s}+\frac{2}{81^s}+\frac{8}{101^s}+\frac{4}{121^s}+\frac{4}{125^s}+\frac{8}{181^s}+\frac{16}{205^s}+\frac{8}{241^s}+\frac{1}{256^s}+\frac{8}{281^s}+\frac{16}{305^s}+\frac{4}{361^s}+\dots$
21	$1+\frac{2}{7^s}+\frac{12}{43^s}+\frac{3}{49^s}+\frac{2}{64^s}+\frac{12}{127^s}+\frac{6}{169^s}+\frac{12}{211^s}+\frac{24}{301^s}+\frac{12}{337^s}+\frac{4}{343^s}+\frac{12}{379^s}+\frac{12}{421^s}+\frac{4}{448^s}+\frac{12}{463^s}+\frac{12}{547^s}+\frac{12}{631^s}+\dots$
24	$1+\frac{1}{4^s}+\frac{2}{9^s}+\frac{1}{16^s}+\frac{4}{25^s}+\frac{2}{36^s}+\frac{4}{49^s}+\frac{1}{64^s}+\frac{8}{73^s}+\frac{3}{81^s}+\frac{8}{97^s}+\frac{4}{100^s}+\frac{4}{121^s}+\frac{2}{144^s}+\frac{4}{169^s}+\frac{8}{193^s}+\frac{4}{196^s}+\frac{8}{225^s}+\dots$
25	$1+\frac{1}{5^s}+\frac{1}{25^s}+\frac{20}{101^s}+\frac{1}{125^s}+\frac{20}{151^s}+\frac{20}{251^s}+\frac{20}{401^s}+\frac{20}{505^s}+\frac{20}{601^s}+\frac{1}{625^s}+\frac{20}{701^s}+\frac{20}{751^s}+\frac{20}{755^s}+\frac{20}{1051^s}+\frac{20}{1151^s}+\frac{20}{1201^s}+\dots$
27	$1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\frac{1}{81^s}+\frac{18}{109^s}+\frac{18}{163^s}+\frac{1}{243^s}+\frac{18}{271^s}+\frac{18}{327^s}+\frac{18}{379^s}+\frac{18}{433^s}+\frac{18}{487^s}+\frac{18}{489^s}+\frac{18}{541^s}+\frac{1}{729^s}+\frac{18}{757^s}+\dots$
28	$1+\frac{2}{8^s}+\frac{12}{29^s}+\frac{1}{49^s}+\frac{3}{64^s}+\frac{12}{113^s}+\frac{6}{169^s}+\frac{12}{197^s}+\frac{24}{232^s}+\frac{12}{281^s}+\frac{12}{337^s}+\frac{2}{392^s}+\frac{12}{421^s}+\frac{12}{449^s}+\frac{4}{512^s}+\frac{12}{617^s}+\frac{12}{673^s}+\dots$
32	$1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{16^s}+\frac{1}{32^s}+\frac{1}{64^s}+\frac{16}{97^s}+\frac{1}{128^s}+\frac{16}{193^s}+\frac{16}{194^s}+\frac{1}{256^s}+\frac{16}{257^s}+\frac{8}{289^s}+\frac{16}{353^s}+\frac{16}{386^s}+\frac{16}{388^s}+\frac{16}{449^s}+\dots$
33	$1+\frac{20}{67^s}+\frac{1}{121^s}+\frac{20}{199^s}+\frac{2}{243^s}+\frac{20}{331^s}+\frac{20}{397^s}+\frac{20}{463^s}+\frac{10}{529^s}+\frac{20}{661^s}+\frac{20}{727^s}+\frac{20}{859^s}+\frac{20}{991^s}+\frac{2}{1024^s}+\frac{20}{1123^s}+\frac{20}{1321^s}+\dots$
35	$1+\frac{24}{71^s}+\frac{24}{211^s}+\frac{24}{281^s}+\frac{24}{421^s}+\frac{24}{491^s}+\frac{24}{631^s}+\frac{24}{701^s}+\frac{12}{841^s}+\frac{24}{911^s}+\frac{24}{1051^s}+\frac{8}{1331^s}+\frac{24}{1471^s}+\frac{12}{1681^s}+\frac{24}{2311^s}+\frac{24}{2381^s}+\dots$
36	$1+\frac{1}{9^s}+\frac{12}{37^s}+\frac{1}{64^s}+\frac{12}{73^s}+\frac{1}{81^s}+\frac{12}{109^s}+\frac{12}{181^s}+\frac{6}{289^s}+\frac{12}{333^s}+\frac{6}{361^s}+\frac{12}{397^s}+\frac{12}{433^s}+\frac{12}{541^s}+\frac{1}{576^s}+\frac{12}{577^s}+\frac{12}{613^s}+\dots$
40	$1+\frac{1}{16^s}+\frac{2}{25^s}+\frac{16}{41^s}+\frac{4}{81^s}+\frac{8}{121^s}+\frac{16}{241^s}+\frac{1}{256^s}+\frac{16}{281^s}+\frac{8}{361^s}+\frac{2}{400^s}+\frac{16}{401^s}+\frac{16}{521^s}+\frac{16}{601^s}+\frac{3}{625^s}+\frac{16}{641^s}+\frac{16}{656^s}+\dots$
44	$1+\frac{20}{89^s}+\frac{1}{121^s}+\frac{20}{353^s}+\frac{20}{397^s}+\frac{10}{529^s}+\frac{20}{617^s}+\frac{20}{661^s}+\frac{20}{881^s}+\frac{20}{1013^s}+\frac{1}{1024^s}+\frac{20}{1277^s}+\frac{20}{1321^s}+\frac{20}{1409^s}+\frac{20}{1453^s}+\frac{10}{1849^s}+\dots$
45	$1+\frac{1}{81^s}+\frac{24}{181^s}+\frac{24}{271^s}+\frac{12}{361^s}+\frac{24}{541^s}+\frac{24}{631^s}+\frac{24}{811^s}+\frac{24}{991^s}+\frac{24}{1171^s}+\frac{24}{1531^s}+\frac{24}{1621^s}+\frac{24}{1801^s}+\frac{24}{2161^s}+\frac{24}{2251^s}+\frac{24}{2341^s}+\dots$
48	$1+\frac{1}{4^s}+\frac{1}{16^s}+\frac{8}{49^s}+\frac{1}{64^s}+\frac{2}{81^s}+\frac{16}{97^s}+\frac{16}{193^s}+\frac{8}{196^s}+\frac{16}{241^s}+\frac{1}{256^s}+\frac{8}{289^s}+\frac{2}{324^s}+\frac{16}{337^s}+\frac{16}{388^s}+\frac{16}{433^s}+\frac{8}{529^s}+\dots$
60	$1+\frac{2}{16^s}+\frac{2}{25^s}+\frac{16}{61^s}+\frac{2}{81^s}+\frac{8}{121^s}+\frac{16}{181^s}+\frac{16}{241^s}+\frac{3}{256^s}+\frac{8}{361^s}+\frac{4}{400^s}+\frac{16}{421^s}+\frac{16}{541^s}+\frac{16}{601^s}+\frac{3}{625^s}+\frac{16}{661^s}+\frac{8}{841^s}+\dots$
84	$1+\frac{2}{49^s}+\frac{2}{64^s}+\frac{12}{169^s}+\frac{24}{337^s}+\frac{24}{421^s}+\frac{24}{673^s}+\frac{2}{729^s}+\frac{24}{757^s}+\frac{12}{841^s}+\frac{24}{1009^s}+\frac{24}{1093^s}+\frac{24}{1429^s}+\frac{24}{1597^s}+\frac{12}{1681^s}+\frac{12}{1849^s}+\dots$

Table 6. Residues of Dedekind zeta functions.

$\phi(n)$	n	R_n	residue	numerical
2	3	1	$\frac{\pi\sqrt{3}}{3^2}$	0.604 600
	4	1	$\frac{\pi}{2^2}$	0.785 398
4	5	$2\log(\frac{1+\sqrt{5}}{2})$	$\frac{2\pi^2\sqrt{5}}{5^3}R_4$	0.339 837
	8	$2\log(1+\sqrt{2})$	$\frac{\pi^2}{2^5}R_8$	0.543 676
	12	$\log(2+\sqrt{3})$	$\frac{\pi^2}{2^2 3^2}R_{12}$	0.361 051
6	7	2.101 819	$\frac{2^2\pi^3\sqrt{7}}{7^4}R_7$	0.287 251
	9	3.397 150	$\frac{2^2\pi^3\sqrt{3}}{3^7}R_9$	0.333 685
8	15	4.661 821	$\frac{2^3\pi^4}{3^3 5^4}R_{15}$	0.215 279
	16	19.534 360	$\frac{\pi^4}{2^{12}}R_{16}$	0.464 557
	20	7.411 242	$\frac{\pi^4}{2^2 5^4}R_{20}$	0.288 769
	24	10.643 594	$\frac{\pi^4}{2^7 3^3}R_{24}$	0.299 995
10	11	26.171 106	$\frac{2^4\pi^5\sqrt{11}}{11^6}R_{11}$	0.239 901
12	13	120.784 031	$\frac{2^5\pi^6\sqrt{13}}{13^7}R_{13}$	0.213 514
	21	70.399 398	$\frac{2^5\pi^6}{3^4 7^6}R_{21}$	0.227 271
	28	123.252 732	$\frac{\pi^6}{2^2 7^6}R_{28}$	0.251 795
	36	162.837 701	$\frac{\pi^6}{2^2 3^{11}}R_{36}$	0.220 933

few cases in Table 6. For $6 \leq \phi(n) \leq 12$, the fundamental units needed for the (numerical) calculation of the regulator were determined by means of the program package KANT [9, 11].

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